Geometry of LPs

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Overview

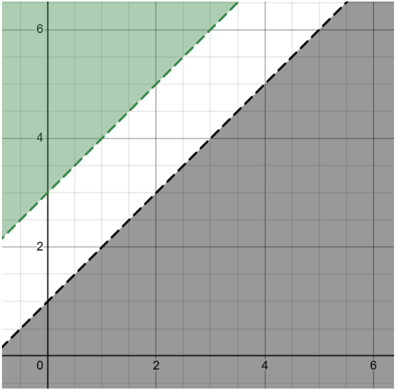
- Limitations on LPs (feasibility, boundedness)
- LP feasible region visualization
- The objective hyperplane
- Convexity of shapes, proof that feasible region of LP is convex
- Number of solutions an LP can have

Limitations on LPs

Not all LPs are solvable.

Example:

$$\begin{aligned} \max \quad & x_1 + x_2 \\ s.t. \quad & x_1 - x_2 \leq -3 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

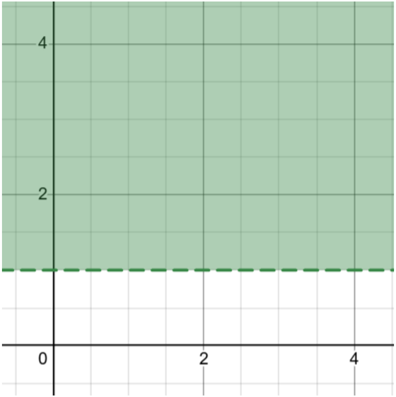


This type of LP is called **infeasible**.

Boundedness

Some LPs aren't possible to solve because the answer would be ∞ or $-\infty$, meaning there is no max or there is no min objective value (and therefore there is no optimum)

$$\begin{aligned} \max \quad & x \\ s.t. \quad & x \geq 1 \\ & x \geq 0 \end{aligned}$$



This type of LP is called **unbounded**.

Feasibility and boundedness

Boundedness is not typically a problem you encounter

- You may see it pop up if you model a problem incorrectly
- Typically an easily fixable mistake, likely a flipped sign

Feasibility is a more common issue

- Difficult to know which constraint is making a problem infeasible
- May need to enable/disable constraints in order to check which one is making the problem infeasible
- ...but sometimes that can cause a problem to become unbounded

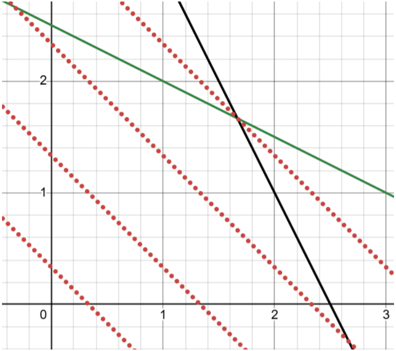
Geometric understanding of LP and objective function

Feasible region is easy enough to reason about in 2 dimensions

- In higher dimensions, not as easy, but can kind of have an idea

Objective function can be easy to reason about as well:

$$
\begin{aligned}
\max \quad & x_1 + x_2 \\
s.t. \quad & x_1 + 2x_2 \leq 5 \\
& 2x_1 + x_2 \leq 5 \\
& x_1, x_2 \geq 0
\end{aligned}
$$



Objective function intuition

Objective function is a **hyperplane** of a lower dimension that we "push through" the feasible region

The last point that the objective plane touches before it exits the feasible region is the **optimal solution** to the LP

Previous example: problem had dimension 2, objective "plane" had dimension 1 (it was a 1-dimensional line)

Can be demonstrated for 3D with tools like GILP:

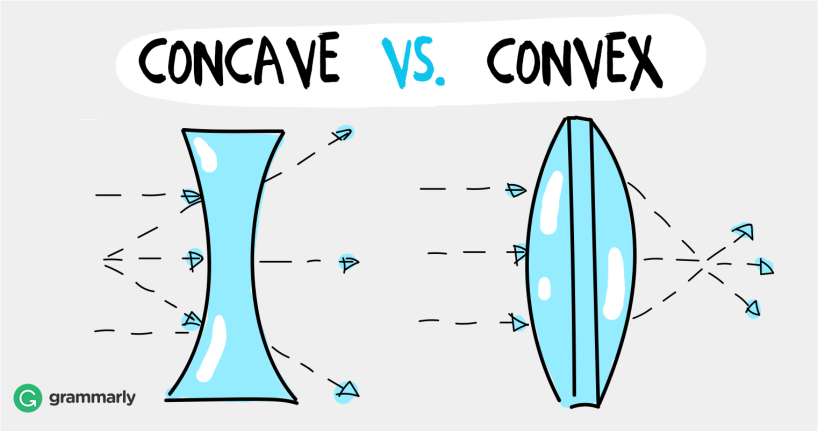
https://gilp.henryrobbins.com/en/latest/examples/3d/

Convexity

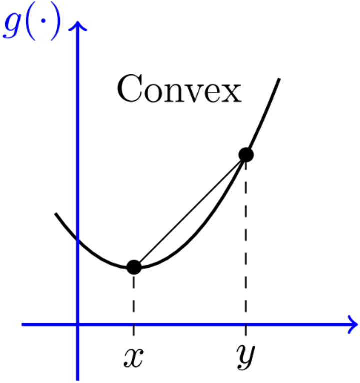
The feasible region of an LP is always *convex*.

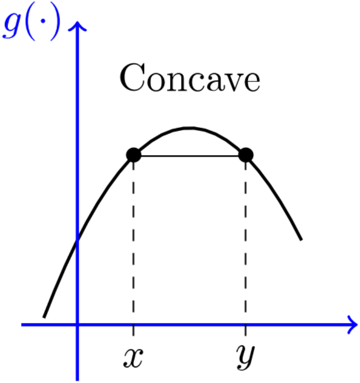
Definition: A **convex** set of points in space is one for which the points on any straight line between any two points are all contained in the set.

Convexity examples (lenses)

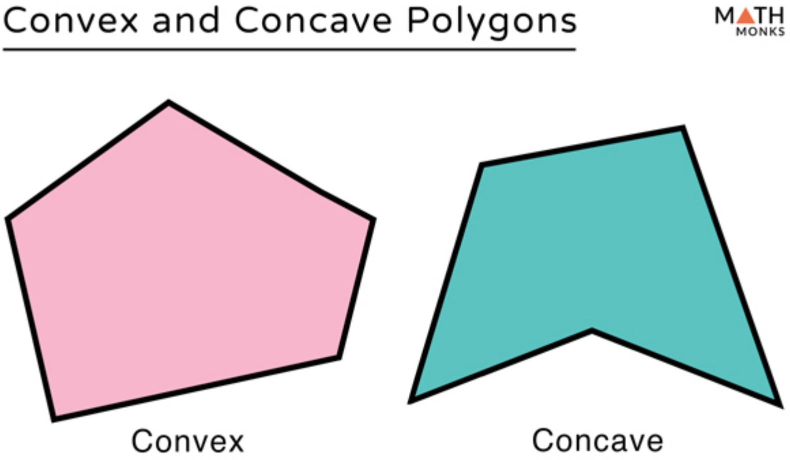


Convexity examples (functions)





Convexity examples (shapes)



This is the one that's most applicable to our understanding of LP feasible regions.

Why is an LP's feasible region convex?

Proof sketch:

1. An LP is composed of a bunch of linear inequalities
2. Each linear inequality divides $\mathbb{R}^n$ into two half-spaces, each of which is convex
3. The intersection of convex sets is convex
4. QED

Convexity and problem difficulty

Why is convexity important?

It makes solving LPs easy:

- Local min $\iff$ global min
- No getting stuck in suboptimal spots
- Makes logic for solving LPs simpler (can find an optimal solution in polynomial time) - no heuristics or error bounds needed

What still ends up being "difficult"

- Finding a point inside the feasible region to start from
- Writing the solvers themselves

LP solution counts

Question: An LP can have:

- A. One solution
- B. Two solutions
- C. Infinitely many solutions
- D. No solutions

Which of these is false?

LP solution counts (cont.)

Question: An LP can have:

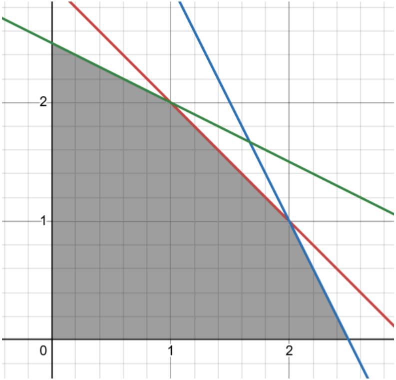
- A. One solution
- ~~B. Two solutions~~ $\leftarrow$ incorrect
- C. Infinitely many solutions
- D. No solutions

An LP cannot have only two solutions.

If there is more than one solution, there must be infinitely many solutions, because the solutions must all lie on the same line.

LP solution count (example)

$$\begin{aligned} \max \quad & x_1 + x_2 \\ s.t. \quad & x_1 + 2x_2 \leq 5 \\ & 2x_1 + x_2 \leq 5 \\ & x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$



There are infinite solutions along the red line, between the red/green and red/blue intersection points