Intro to linear programming

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Overview

- What is linear programming?
- Example
- Motivation for general solution

What is linear programming?

NOT “programming” in the typical sense

- More related to scheduling (like TV programming)

A way to model problems as linear inequalities - Efficiently solvable by many tools: - Python packages, many backend solvers in C++, even MS Excel

Example

$$\max \quad x_1 + x_2$$

$$s.t. \quad x_1 \leq 3$$

$$x_2 \leq 4$$

$\min$ / $\max \; -x_1 - x_2$, $c^T x$, $c = \langle -1, -1 \rangle$

$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$x_1 \geq 3$

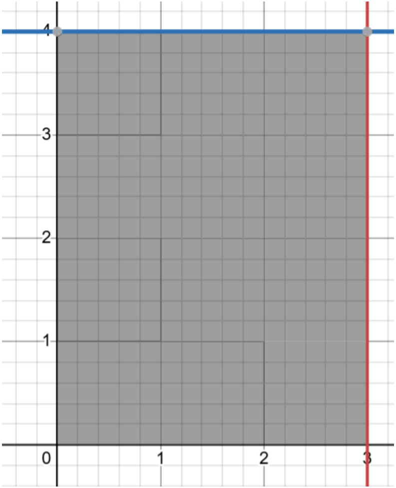
$-x_1 \leq -3$

$x_1, x_2 \geq 0$

II | I
III | IV

Example (cont.)

$$\begin{aligned} \max \quad & x_1 + x_2 \\ s.t. \quad & x_1 \leq 3 \\ & x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$



More complexity

Not all linear programs are this easy to solve:

$$
\begin{aligned}
\max \quad & x_1 + x_2 + \cdots + x_n \\
s.t. \quad & x_1 \leq 3 \\
& x_2 + x_3 \leq 4 \\
& x_2 + x_3 - x_5 \leq 4 \\
& x_5 + x_1 - x_9 \geq 50 \\
& \vdots \\
& x_1, x_2, \ldots, x_n \geq 0
\end{aligned}
$$

Real-world usecases for linear programming

- Airline scheduling
- Efficient redistribution of citibikes
- Portfolio optimization
- Petroleum blending
- Food ingredient mixing
- Reach maximization for advertisements
- Minimizing waste in factory settings
- Routing delivery vehicles (e.g., UberEats or Doordash)

These problems are not so simple!

Example: Meal delivery routing problem (2024, Kancharla, S.R., Van Woensel, T., Waller, S.T. et al.)

$$\max \sum_{i\in T_d} p_i - \sum_{i,j\in T} c_{ij} - \alpha\left(\sum_{i\in T_d} \mu_i - E[Q(x,\xi)]\right)$$

$$p_i \le w_i \sum_{j\in N}\sum_{k\in V} x_{ji}^k \quad \forall i \in T_d$$

$$c_{ij} \ge \sum_{k\in V} m_k x_{ij}^k t_{ij} \quad \forall i,j \in T, i \ne j$$

$$\sum_{k\in V}\left(\sum_{j\in T\setminus\{i\}} x_{ji}^k\right) \le 1 \quad \forall i \in T$$

$$\sum_{k\in V}\left(\sum_{j\in T\setminus\{i\}} x_{ij}^k\right) = \sum_{k\in V}\left(\sum_{j\in T\setminus\{i\}} x_{ji}^k\right) \quad \forall i \in T$$

Average LP in a published paper...

$$\sum_{j\in T}\sum_{i\in T_p} x_{ij}^k = \sum_{j\in T}\sum_{i\in T_d} x_{ji}^k \quad \forall k \in V, i \ne j$$

$$\sum_{j\in N\setminus\{i\}}\sum_{k\in V} a_{ji}^k \le \sum_{j\in N\setminus\{i\}}\sum_{k\in V} a_{ij}^k - (s_i + t_{ij}) x_{ij}^k \quad \forall i \in T$$

$$e_i x_{ij}^k \le a_{ij}^k \quad \forall k \in V, i \in T, j \in T\setminus\{i\}$$

$$e_k x_{ij}^k \le a_{ij}^k \le l_k x_{ij}^k \quad \forall k \in V, i \in N, j \in N\setminus\{i\}$$

$$\sum_{j\in N\setminus\{i\}} a_{ji}^k \le \sum_{j\in T\setminus\{i+n\}} a_{ji+n}^k \quad \forall k \in V, i \in P$$

$$\sum_{k\in V}\sum_{j\in N\setminus\{i\}} y_{ij}^k - d_i x_{ij}^k = \sum_{k\in V}\sum_{j\in N\setminus\{i\}} y_{ji}^k \quad \forall i \in T$$

$$y_{ij}^k \le \beta_k x_{ij}^k \quad \forall k \in V, i \in N, j \in N\setminus\{i\}$$

$$\mu_i + l_i \ge \sum_{k\in V}\sum_{j\in N} a_{ji}^k + s_i x_{ij}^k \quad \forall i \in T_d$$

$$x_{ij}^k \in \{0,1\} \quad \forall k \in V, i \in N, j \in N$$

$$y_{ij}^k \ge 0, a_{ij}^k \ge 0, p_i \ge 0, c_{ij} \ge 0 \quad \forall k \in V, i \in N, j \in N$$

Canonical form

Need to make things simpler to reason about them

Canonical form:

$$\begin{aligned} \max \quad & c^{\mathsf{T}} x \\ s.t. \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

- A is a matrix
- x is now a vector of decision variables
- c is a (row) vector (so c^{T} is a column vector)
- b is a (column) vector

References

Kancharla, S.R., Van Woensel, T., Waller, S.T. et al. Meal Delivery Routing Problem with Stochastic Meal Preparation Times and Customer Locations. Netw Spat Econ 24, 997–1020 (2024). https://doi.org/10.1007/s11067-024-09643-1