Linear algebra review

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Linear algebra topics

- Vectors
  - Adding, multiplying
  - Dot product
  - Linear independence
  - What $\mathbb{R}^n$ is
- Matrices
  - Multiplication, inverse, transpose
  - Row reduction

Vectors

Row vector:

$$[1, 2, 3]$$

Column vector:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Adding and multiplying vectors

Adding vectors:

$$[1, 2, 3] + [4, 5, 6] = [5, 7, 9]$$

Scalar multiplication:

$$3 \cdot [1, 2, 3] = [3, 6, 9]$$

Dot product

Dot product requires a row vector and a column vector:

$$[1, 2, 3] \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = (1 \cdot 4) + (2 \cdot 5) + (3 \cdot 6) = 4 + 10 + 18 = 32$$

Linear independence

A set of n vectors are linearly independent if it's not possible to use any combination of $n-1$ (or fewer) vectors to get the remaining one as a result.

Linearly independent:

$$\{[1,0,0],[0,1,0]\}$$

Linearly dependent (the first two are scalar multiples of each other):

$$\{[2,4,6],[4,8,12],[1,2,5]\}$$

Also linearly dependent (not as obvious):

$$\{[1,2,3],[3,5,6],[2,2,0]\}$$

Above, $4 \cdot [1,2,3] - 2 \cdot [3,5,6] = [2,2,0]$.

Vector spaces

The set of real numbers is denoted by $\mathbb{R}$. This includes everything like π, e, 1, 432857239.3958593, -32789.4, etc.

The only vector space we care about is $\mathbb{R}^n$.

A vector is in $\mathbb{R}^n$ if it has n entries and each entry is in $\mathbb{R}$.

Matrices

A matrix is an ordered list of vectors.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Whether the vectors that comprise a matrix are row vectors (e.g., $[1, 2, 3]$) or column vectors (e.g., $[1, 4, 7]$) doesn't really matter.

We can extract elements of a matrix using both the row and column index, like this:

$$A_{1,1} = 1, A_{2,3} = 6, A_{3,2} = 8$$

Matrix dimensions

A matrix could have different numbers of rows and columns:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Above, A is a 2×3 matrix. When indexing and when describing dimensions, rows always come before columns.

A matrix with dimensions $n \times m$ is in $\mathbb{R}^{m \times n}$, so $A \in \mathbb{R}^{2 \times 3}$.

Matrix multiplication

Let A be a matrix with dimension $m \times n$ and let B be a matrix with dimension $p \times q$.

$$AB \implies (m \times \boxed{n) \cdot (p} \times q) \implies n \text{ and } p \text{ need to match}$$

$$BA \implies (p \times \boxed{q) \cdot (m} \times n) \implies q \text{ and } m \text{ need to match}$$

The resulting matrix will have dimension $m \times q$.

Matrix multiplication example

The entries of the resulting matrix AB will be the result of taking the dot product of rows from A and columns from B:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} [1,2,3] \cdot [7,9,11] & [1,2,3] \cdot [8,10,12] \\ [4,5,6] \cdot [7,9,11] & [4,5,6] \cdot [8,10,12] \end{bmatrix}$$

$$= \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

Identity matrix

The identity matrix I_n is the n-dimensional matrix that has ones on the diagonal and zeros everywhere else.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transpose

The **transpose** of a matrix is what you get when you “flip” the matrix over the diagonal. This means you swap the rows and columns.

We denote the transpose of a matrix A by $A^\top$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$A^\top = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Solving a matrix-vector equation

We will frequently see the following equation, where A is a matrix and b is a vector:

$$Ax = b$$

Here, the goal is to find what x must be to satisfy this equation.

The dimensions of x and b must match, and both must have length equal to n, where A is an $n \times n$ matrix.

Row reduction

If we are given A and b as follows:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 3 \\ 2 & 5 & 9 \end{bmatrix}, b = \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}$$

We first *augment* A with b like so:

$$[A \mid b] = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 5 & 4 & 3 & 7 \\ 2 & 5 & 9 & 8 \end{bmatrix}$$

We can then *row-reduce* $[A \mid b]$ to get the following:

$$\text{rref}([A \mid b]) = \begin{bmatrix} 1 & 0 & 0 & -19/2 \\ 0 & 1 & 0 & 39/2 \\ 0 & 0 & 1 & -47/6 \end{bmatrix}$$

Row reduction in practice

```
import numpy as np
from scipy import linalg

A = np.array([[1, 2, 3], [5, 4, 3], [2, 5, 9]])
b = np.array([6, 7, 8])

x = linalg.solve(A, b)
print(x)
```

```
[-9.5        19.5        -7.83333333]
```

Inverting a matrix

To find the inverse of a matrix, augment it with the identity matrix I_n and row-reduce it.

For example, let's row-reduce the matrix we had from before:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 3 \\ 2 & 5 & 9 \end{bmatrix}$$

First, augment it with I_3 to get $[A \mid I_3]$:

$$[A \mid I_3] = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 5 & 4 & 3 & 0 & 1 & 0 \\ 2 & 5 & 9 & 0 & 0 & 1 \end{bmatrix}$$

Now, we can row-reduce $[A \mid I_3]$:

$$\text{rref}\left(\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 5 & 4 & 3 & 0 & 1 & 0 \\ 2 & 5 & 9 & 0 & 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -7/2 & 1/2 & 1 \\ 0 & 1 & 0 & 13/2 & -1/2 & -2 \\ 0 & 0 & 1 & -17/6 & 1/6 & 1 \end{bmatrix}$$

Matrix inversion in practice

```
import numpy as np
from scipy import linalg

A = np.array([[1, 2, 3], [5, 4, 3], [2, 5, 9]])

A_inv = linalg.inv(A)
print(A_inv)
```

```
[[-3.5         0.5         1.        ]
 [ 6.5        -0.5        -2.        ]
 [-2.83333333  0.16666667  1.        ]]
```